LIII. *The Reſolution of a General Propoſition for Determining the* horary *Alteration of the Poſition of the Terreſtrial Equator, from the Attraction of the Sun and Moon: With ſome Remarks on the Solutions given by other Authors to that difficult and important Problem. By Mr.* Tho. Simpſon, *F. R. S*

Read Dec. 22, 1757.

SINCE the time, that that excellent Aſtronomer, my much honoured friend Dr. Bradley, publiſhed his obſervations and diſcoveries concerning the inequalities of the preceſſion of the equinox, and of the obliquity of the ecliptic, depending on the poſition of the lunar nodes, mathematicians, in different parts of Europe, have ſet themſelves diligently to compute, from phyſical principles, the effects produced by the ſun and moon, in the poſition of the terreſtrial equator; and to examine whether theſe effects do really correſpond with the obſervations

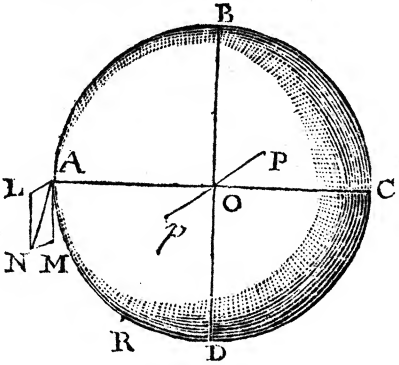
Two papers on this ſubject have already appeared in the Philoſophical Tranſactions; in which the authors have ſhewn evident marks of ſkill and penetration. There is, nevertheleſs, one part of the ſubject, that ſeems to have been paſſed over without a due degree of attention, as well by both thoſe gentlemen, as by Sir Iſaac Newton himſelf.

This part, which, upon account of phyſical difficulties, is indeed ſomewhat ſlippery and perplexing, I ſhall make the principal ſubject of this eſſay.

GENERAL PROPOSITION.

Suppoſing an homogeneous ſphere OABCD (Fig. 1.) *revolving uniformly about its centre, to be acted on at the extremity* A *of the radius* OA, *in a direction* AL *perpendicular to the plane of the equator* ABCD, *and parallel to the axis of rotation* Pp, *by a given force, tending to generate a new motion of rotation at right angles to the former; It is propoſed to determine the change, that will ariſe in the direction of the rotation in conſequence of the ſaid force.*

FIG. 1.



Let *F* denote the given force, whereby the motion about the axis P*p* is diſturbed, ſuppoſing *f* to repreſent the centrifugal force of a ſmall particle of matter in the circumference of the equator, ariſing from the ſphere's rotation; and let the whole number of ſuch particles, or the content of the ſphere, be denoted by *c*: let alſo the momentum of rotation of the whole ſphere, or of all the particles, be ſuppoſed, in proportion to the momentum of an equal number of particles, revolving at the diſtance O A of the remoteſt point A, as n is to *unity*.

It is well known, that the centripetal force, whereby any body is made to revolve in the circumference of a circle, is ſuch, as is ſufficient to generate all the motion in the body, in a time equal to *that*, wherein the body deſcribes an arch of the circumference, equal in length to the radius. Therefore, if we here take the arch $AR = OA$, and aſſume m to expreſs the time, in which that arch would be uniformly deſcribed by the point A, the *motion* of a particle of matter at A (whoſe central force is repreſented by f) will be equal to *that*, which might be uniformly generated by the force f, in the time m; and the motion of as many particles (revolving, all, at the ſame diſtance) as are expreſſed by cn (which, by hypotheſis, is equal to the momentum of the whole body), will, conſequently, be equal to the momentum, that might be generated by the force $f \times cn$, in the ſame time m. Whence it appears, that the momentum of the whole body about its axe Pp is in proportion to the momentum generated in a given particle of time m', by the given force F in the direction A L, as $ncf \times m$ is to $F \times m'$, or, as *unity* to $\frac{F}{ncf} \times \frac{m'}{m}$ (becauſe the quantities of motion produced by unequal forces, in unequal times, are in the ratio of the forces and of the times, conjunctly). Let, therefore, A L be taken in proportion to A M, as $\frac{F}{ncf} \times \frac{m'}{m}$ is to *unity* (ſuppoſing A M to be a tangent to the circle A B C D in A), and let the parallelogram A M N L be compleated; drawing alſo the diagonal A N; then, by

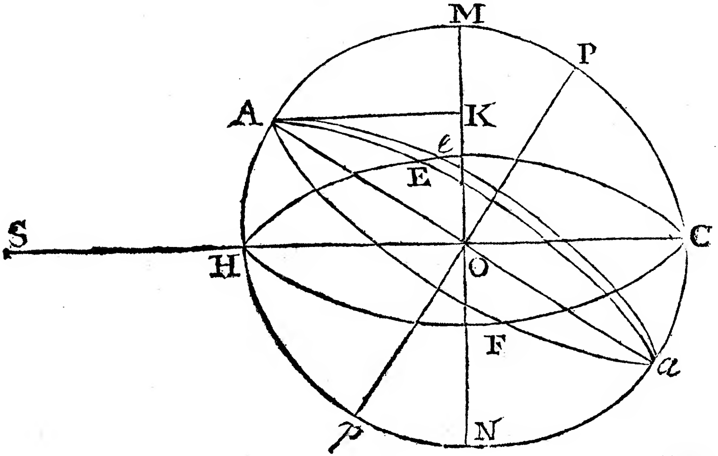
the

the compoſition of forces, the angle N A M (whoſe tangent, to the radius O A, is expreſſed by O A × $\frac{F}{ncf} \times \frac{m'}{m}$) will be the change of the direction of the rotation, at the end of the aforeſaid time (m'). But, this angle being exceeding ſmall, the tangent may be taken to repreſent the meaſure of the angle itſelf; and, if Z be aſſumed to repreſent the arch deſcribed by A, in the ſame time (m') about the center O, we ſhall alſo have $\frac{m'}{m} = \frac{Z}{AR} = \frac{Z}{AO}$, and conſequently $OA \times \frac{F}{ncf} \times \frac{m'}{m} = Z \times \frac{F}{ncf}$. From whence it appears, that the angle expreſſing the change of the direction of the rotation, during any ſmall particle of time, will be in proportion to the angle deſcribed about the axe of rotation in the ſame time, as $\frac{F}{ncf}$ is to *unity. Q. E. I.*

Altho', in the preceding propoſition, the body is ſuppoſed to be a perfect ſphere, the ſolution, nevertheleſs, holds equally true in every other ſpecies of figures, as is manifeſt from the inveſtigation. It is true, indeed, that the value of n will not be the ſame in theſe caſes, even ſuppoſing thoſe of c, f and F to remain unchanged; except in the ſpheroid only, where, as well as in the ſphere, n will be $= \frac{2}{5}$; the momentum of any ſpheroid about its axis being 2-5ths of the momentum of an equal quantity of matter placed in the circumference of the equator, as is very eaſy to demonſtrate.

But to ſhew now the uſe and application of the general proportion here derived, in determining the regreſs of the equinoctial points of the terreſtrial ſpheroid, let A E *a* F (*Fig.* 2.) be the equator, and P *p* the axis of the ſpheroid: alſo let H E C F repreſent the plane of the ecliptic, S the place of the ſun, and H A P N H the plane of the ſun's declination, making right-angles with the plane of the equator A E *a* F: then, if AK be ſuppoſed parallel, and O K M perpendicular, to O S, and there be aſſumed T and t to expreſs the reſpective times of the annual and diurnal revolutions of the earth, it will appear (from the *Principia*, B. III. prop. xxv.) that the force, with which a particle of matter at A tends to recede from the line O M in conſequence of the ſun's attraction, will be expreſſed by $\frac{3tt}{TT}\times\frac{AK}{OA}\times f$; f denoting the centrifugal force of the ſame particle, ariſing from the diurnal rotation. Hence, by the reſolution of forces, $\frac{3tt}{TT}\times\frac{AK}{OA}\times\frac{OK}{OA}\times f$ will be the effect of that particle, in a direction perpendicular to O A, to turn the earth about its center O.

FIG. 2.



But it is demonſtrated by Sir Iſaac Newton, and by other authors, that the force of all the particles, or of all the matter in the whole ſpheroid A P ap, to turn *it* about its center, is equal to $\frac{1}{5}$th of the force of a quantity of matter, placed at A, equal to the exceſs of the matter in the whole ſpheroid above *that* in the inſcribed ſphere, whoſe axis is Pp. Now this exceſs (aſſuming the ratio of π to 1, to expreſs *that* of the area of a circle to the ſquare of the radius) will be truly repreſented by $\frac{4\pi}{3} \times OP \times \overline{OA^2 - OP^2}$; and, conſequently, the force of all the matter in the whole earth, by $\frac{3tt}{TT} \times \frac{AK}{OA} \times \frac{OK}{OA} \times \frac{4\pi}{15} \times OP \times \overline{OA^2 - OP^2}$. Let, therefore, this quantity be now ſubſtituted for F, in the general formula $\frac{F}{ncf}$, writing, at the ſame time, $\frac{4\pi}{3} \times OA^2 \times OP$, and $\frac{2}{5}$, in the place of their equals c and n; by which means we have (here) $\frac{F}{ncf} = \frac{3tt}{2TT} \times \frac{OA^2 - OP^2}{OA^2} \times \frac{AK \times OK}{OA^2}$. Put the given quantity $\frac{3tt}{2TT} \times \frac{OA^2 - OP^2}{OA^2} = k$; and let the angle E A e repreſent the horary alteration of the poſition of the terreſtrial equator, ariſing from the force F (here determined), and let the arch Ee be the regreſs of the equinoctial point E, correſponding thereto: then, in the triangle E A e (conſidered as ſpherical) it will be ſin. e : ſin. AE (: : ſin. E Ae : ſin. Ee) : : E Ae : Ee $\left(= \frac{\text{ſin. AE} \times \text{EA}e}{\text{ſin. E}}\right) = k \times \frac{\text{ſin. AE}}{\text{ſin. E}} \times \frac{AK \times OK}{OA^2} = k \times \frac{\text{ſin. AE} \times \text{coſ. AH} \times \text{ſin. AH}}{\text{ſin. E}}$. But in the triangle E H A, right-angled at A (where H A is ſuppoſed

ſuppoſed to repreſent the ſun's declination, AE his right aſcenſion, and HE his diſtance from the equinoctial point E *) we have *(per ſpherics)*

$$\text{ſin. AE} : 1\ (\text{rad.}) :: \text{co-t. E} : \text{co-t. AH},$$

$$\overline{\text{ſin. AH}}^2 : \overline{\text{ſin. EH}}^2 :: \overline{\text{ſin. E}}^2 : 1^2 (\overline{\text{rad.}^2})$$

From whence we get, $\text{ſin. AE} \times \text{co-t. AH} \times \overline{\text{ſin. AH}}^2 = \overline{\text{ſin. EH}}^2 \times \text{co-t. E} \times \overline{\text{ſin. E}}^2$. But $\text{co-t. AH} \times \text{ſin. AH} = \text{co-ſ. AH} \times 1\ (\text{rad.})$, and $\text{co-t. E} \times \text{ſin. E} = \text{co-ſ. E} \times 1\ (\text{rad.})$: therefore $\text{ſin. AE} \times \text{co-ſ. AH} \times \text{ſin. AH} = \overline{\text{ſin. EH}}^2 \times \text{co-ſ. E} \times \text{ſin. E}$; and, conſequently, $k \times \frac{\text{ſin. AE} \times \text{co-ſ. AH} \times \text{ſin. AH}}{\text{ſin. E}} = k \times \text{co-ſ. E} \times \overline{\text{ſin. EH}}^2$ $(= \text{E}e)$.

Let, now, the ſun's longitude EH be denoted by Z (conſidered as a flowing quantity); then, $\overline{\text{ſin. Z}}^2$ being $= \frac{1}{2} - \frac{1}{2}\ \text{co-ſ. 2 Z}$, we ſhall have $k \times \text{co-ſ. E} \times \overline{\text{ſin. EH}}^2 = \frac{1}{2} k \times \text{co-ſ. E} \times \overline{1 - \text{co-ſ. 2 Z}}$. But the angle deſcribed about the axe of rotation P*p*, in the time that the ſun's longitude is augmented by the particle $\dot{Z}$, will be $= \frac{T}{t} \times \dot{Z}$. Therefore (by the general propoſition) we have, as $1 : \frac{1}{2} k \times \text{co-ſ. E} \times \overline{1 - \text{co-ſ. 2 Z}} :: \frac{T}{t} \times \dot{Z} : \frac{1}{2} k \times \frac{T}{t} \times \text{co-ſ. E} \times \overline{\dot{Z} - \dot{Z}\ \text{co-ſ. 2 Z}}$, the true regreſs of the equinoctial point E, during that time: whoſe fluent, $\frac{1}{2}k \times \frac{T}{t} \times \text{co-ſ.}\,E \times \overline{Z - \frac{1}{2}\,\text{ſin.}\,2\,Z}$, will conſequently be the total regreſs of the point E, in the time that the ſun, by his apparent motion, deſcribes the arch HE or Z; which, on the ſun's arrival at the ſolſtice, becomes barely $= \frac{1}{2}k \times \frac{T}{t} \times \text{co-ſ.}\,E \times$ an arch of 90°: the quadruple whereof, or $\frac{1}{2}k \times \frac{T}{t} \times \text{co-ſ.}\,E \times 360^\circ$ $\left(= \frac{3t}{4T} \times \frac{OA^2 - OP^2}{OA^2} \times \text{co-ſ.}\,E \times 360^\circ\right)$ is therefore the whole annual preceſſion of the equinox cauſed by the ſun. This, in numbers (taking $\frac{OP}{OA} = \frac{229}{230}$) comes out $\frac{3}{4 \times 366\frac{1}{4}} \times \frac{2}{230\frac{1}{2}} \times 0.917176 \times 360^\circ = 21'' \, 6'''$.

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* No error ariſes from conſidering the triangles EA*e* and AEH, as being formed on the ſurface of a ſphere, tho' the earth itſelf is not accurately ſuch. The angle (EA*a*) repreſenting the effect of the ſolar force, is properly referred to the ſurface of a ſphere; therefore (after the meaſure thereof is truly determined) the figure AP*ap* is itſelf taken as a ſphere, in order to avoid the trouble of introducing a new ſcheme.

The very ingenious M. Silvabelle, in his eſſay on this ſubject, inſerted in the 48th volume of the Philoſophical Tranſactions, makes the quantity of the annual preceſſion of the equinox, cauſed by the ſun, to be the half, only, of what is here determined. But this gentleman appears to have fallen into a twofold miſtake. Firſt, in finding the *momenta of rotation* of the terreſtrial ſpheroid, and of a very ſlender ring, at the equator thereof; which *momenta* he refers to an axis perpendicular to the plane of the ſun's declination, inſtead of the proper axe of rotation, ſtanding at right angles to the plane of the equator. The difference, indeed, ariſing from thence, with reſpect to the ſpheroid (by reaſon of its near approach to a ſphere) will be inconſiderable; but, in the ring, the caſe will be quite otherwiſe; the equinoctial points thereof being made to recede juſt twice as faſt as they ought to do. This may ſeem the more ſtrange, if regard be had to the concluſions, relating to the nodes of a ſatellite, derived from this very aſſumption. But, that theſe concluſions are true, is owing to a ſecond, or ſubſequent miſtake, at Art. 27; where the meaſure of the ſun's force is taken the half, only, of the true value; by means whereof the motion of the equinoctial points of the ring is reduced to its proper quantity, and the motion of the equinoctial points of the terreſtrial ſpheroid, to the half of what it ought to be.

That expert geometrician M. Cha. Walmſley, in his Eſſay on the Preceſſion of the Equinox, printed in the laſt volume of the Philoſophical Tranſactions, has judiciouſly avoided all miſtakes of this laſt kind, reſpecting the ſun's force, by purſuing the method, pointed out by Sir Iſaac Newton; but, in determining the effect of that force, has fallen into others, not leſs conſiderable than thoſe above adverted to.

In his third Lemma, the momentum of the whole Earth, about its diameter, is computed on a ſuppoſition, that the momentum or force of each particle is proportional to its diſtance from the axis of motion, or barely as the quantity of motion in ſuch particle, conſidered abſtractedly. No regard is, therefore, had to the lengths of the unequal levers, whereby the particles are ſuppoſed to receive and communicate their motion: which, without doubt, ought to have been included in the conſideration.

In his firſt propoſition, he determines, in a very ingenious and conciſe manner, the true annual motion of the nodes of a ring (or of a ſingle ſatellite) at the earth's equator, revolving with the earth itſelf, about its center, in the time of one ſiderial day. This motion he finds to be $=\frac{3\ \text{co-ſ.}\ 23^\circ 29'}{4\ \text{rad.}}\times\frac{1}{366\frac{1}{4}}\times 360^\circ$. Then, in order to infer from thence, the motion of the equinoctial points of the earth itſelf, he, firſt, diminiſhes that quantity, in the ratio of 2 to 5: Becauſe (as is demonſtrated by Sir Iſaac Newton in his 2d Lemma) the whole force of all the particles ſituated without the ſurface of a ſphere, inſcribed in the ſpheroid, to turn the body about its center, will be only 2-5ths of the force of an equal number of particles uniformly diſpoſed round the whole circumference of the equator, in the faſhion of a ring. The quantity $\left(\frac{3\ \text{co-ſ.}\ 23^\circ 29'}{4\ \text{rad.}}\times\frac{2}{5}\times\frac{1}{366\frac{1}{4}}\times 360^\circ\right)$ thus ariſing, will, therefore, expreſs the true motion of the equinoctial points of a ring, equal in quantity of matter to the exceſs of the whole earth above the inſcribed ſphere, when the force whereby the ring tends to turn about its diameter is ſuppoſed equal to the force whereby the earth itſelf tends to turn about the ſame diameter, in conſequence of the ſun's attraction. Thus far our author agrees with Sir Iſaac Newton; but, in deriving from hence the motion of the equinoctial points of the earth itſelf, he differs from him; and, in the corollary to his third Lemma, aſſigns the reaſons, why he thinks Sir Iſaac Newton, in this particular, has *wandered a little from the truth.* Inſtead of diminiſhing the quantity above exhibited (as Sir Iſaac has done) in the ratio of all the motion in the ring to the motion in the whole earth, he diminiſhes it in the ratio of the motion of all the matter above the ſurface of the inſcribed ſphere to the motion of the whole earth: which matter, tho' equal to that of the ring, has nevertheleſs a different momentum, ariſing from the different ſituation of the particles in reſpect to the axis of motion.

But ſince the aforeſaid quantity, from whence the motion of the earth's equinox is derived, as well by this gentleman, as by Sir Iſaac Newton, expreſſes truly the annual regreſs of the equinoctial points of the ring (and not of the hollow figure formed by the ſaid matter, which is greater, in the ratio of 5 to 4) it ſeems, at leaſt, as reaſonable to ſuppoſe, that the ſaid quantity, to obtain from thence the true regreſs of the equinoctial points of the earth, ought to be diminiſhed in the former of the two ratios above ſpecified, as that it ſhould be diminiſhed in the latter. But, indeed, both theſe ways are defective, even ſuppoſing the momenta to have been truly computed; the ratio, that ought to be uſed here, being that of the momenta of the ring and earth about the proper axe of rotation of the two figures, ſtanding at right-angles to the plane of the ring and of the equator. Now this ratio, by a very eaſy computation, is found to be as $\overline{230}^2 - \overline{229}^2$ to $\frac{2}{5}$ of $\overline{230}^2$; whence the quantity ſought comes out $= \frac{3 \text{ co-ſ. } 23^\circ\, 29'}{4 \text{ rad.}} \times \frac{1}{366\frac{1}{4}} \times \frac{\overline{230}^2 - \overline{229}^2}{\overline{230}^2} \times 360^\circ = 21''\, 6'''$: which is the ſame that we before found it to be, and the double of what this author makes it.

What has been ſaid hitherto, relates to that part of the motion only, ariſing from the force of the ſun. It will be but juſtice to obſerve here, that the effect of the moon, and the inequalities depending on the poſition of her nodes, are truly aſſigned by both the gentlemen

gentlemen above-named; the ratio of the diameters of the earth, and the denſity of the moon being ſo aſſumed, as to give the maxima of thoſe inequalities, ſuch as the obſervations require: in conſequence whereof, and from the law of the increaſe and decreaſe (which is rightly determined by theory, tho' the abſolute quantity is not) a true ſolution, in every other circumſtance, is obtained.

The freedom, with which I have expreſſed myſelf, and the liberty I have here taken, to animadvert on the works of men, who, in many places, have given inconteſtible proofs of ſkill and genius, may, I fear, ſtand in need of ſome apology. 'Tis poſſible I may be thought too peremptory. Indeed, I might have delivered my ſentiments with more caution and addreſs: but, had not I imagined myſelf quite clear in what has been advanced, from a multitude of concurrent reaſons, I ſhould have thought it too great a preſumption to have ſaid any thing at all here, on this ſubject. The great regard I have for this Society, of which I have the honour to be a member, will, I hope, be conſidered as the motive for my having attempted to rectify ſome overſights, that have occurred in the works of this learned body.